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Klaus Altmann · Duco van Straten

Smoothing of quiver varieties

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Abstract. We show that Gorenstein singularities that are cones over singular Fano varieties provided by so-called flag quivers are smoothable in codimension three. Moreover, we give a precise characterization about the smoothability in codimension three of the Fano variety itself.

1. Introduction

(1.1) Quivers and varieties of quiver representations appear in various places in mathematics, see for example [1]. In [7] it was shown that grassmanians and partial flag manifolds have a toric degeneration that can be described by a certain quiver. This type of quivers can be generalised to what we call *flag quivers*.

We show in this paper that toric Gorenstein singularities X provided by such flag quivers Q are smoothable in codimension three, cf. Corollary 33.

The idea is to determine their infinitesimal deformation spaces $T^k_X (k = 1, 2)$: T^1_X describes the infinitesimal deformations, and T^2_X contains the obstructions for extending deformations to larger base spaces—see [11] for more details. The results will show that all deformations are unobstructed (cf. Theorem 32) and, moreover, that there are enough of them for providing a smoothing in codimension three (cf. Theorem 29).

The singularities X turn out to be cones over singular Fano varieties $\mathbb{P}_{\nabla(Q)}$. In Theorem 31, we determine the (embedded) infinitesimal deformations of these projective varieties. This yields to a precise characterization of those flag quivers Q leading to Fanos $\mathbb{P}_{\nabla(Q)}$ which are smoothable in codimension three. The condition is that every so-called simple knot (cf. Definition 26) can be by-passed with a multipath connecting its neighbors, cf. Corollary 33 again.

The deformation theory of three-codimensional singularities in toric Fano varieties becomes important if one considers three-dimensional Calabi-Yau subvarieties given by nef partitions of the defining polytopes, cf. [4–6].

K. Altmann (✉): Fachbereich Mathematik und Informatik, Freie Universität Berlin, Arnimalle 3, 14195 Berlin, Germany. e-mail: altmann@math.fu-berlin.de

D. van Straten: Fachbereich Mathematik (17), Johannes Gutenberg-Universität, 55099 Mainz, Germany. e-mail: straten@mathematik.uni-mainz.de

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(1.2) The paper is organized as follows: In Sect. 2 we recall the main facts about affine, toric Gorenstein singularities X and their infinitesimal deformation theory. In particular, we show how the modules T^k_X are linked to certain invariants $D^k(\Delta)$ of the polytopes Δ defining X.

In Sect. 3, we study mutually dual polytopes $\nabla(Q)$ and $\Delta(Q)$ provided by quivers, i.e. oriented graphs Q. In the smooth case, i.e. if $\nabla(Q)$ looks like an orthant in a neighborhood of every vertex, this has already been done in [1] to describe the moduli spaces of representations of the quiver Q. Here, we focus on the singular case.

An important technical term is that of the tightness of a quiver Q. This will be studied in Sect. 4. Every quiver can be "tightened", and this property straightens the relation between Q ant its associated polytopes. In particular, it allows to determine all faces of $\Delta(Q)$ with a non-trivial D^1-invariant. Then, we restrict ourselves to the special case of so-called flag quivers. They are introduced (and their name is explained) in Sect. 5. Moreover, we determine their Picard group.

Eventually, Sect. 6 deals with the deformation theory of flag quivers. For them, it is possible to apply Theorems 1 and 2, i.e. we can detect all non-vanishing D^1, but prove the vanishing of D^2 for all relevant faces of $\Delta(Q)$. This translates into the lack of obstructions for our singularities X.

2. Toric Gorenstein singularities

(2.1) Let N, M be two mutually dual free abelian groups of finite rank; denote by $N_{\mathbb{R}}$, $M_{\mathbb{R}}$ the vector spaces obtained by extending the scalars. Each polyhedral, rational cone $\sigma \subseteq N_{\mathbb{R}}$ with apex in 0 gives rise to an affine, so-called toric variety $\mathbb{TV}(\sigma, N) := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. See [8,9] for more details.

The toric variety $\mathbb{TV}(\sigma, N)$ is Gorenstein if and only if σ is the cone over a lattice polytope Δ sitting in an affine hyperplane of height one in $N_{\mathbb{R}}$, i.e. if there is a primitive $R^* \in M$ such that $\Delta \subseteq [R^* = 1] \subseteq N_{\mathbb{R}}$. In this situation, we denote $X_\Delta := \mathbb{TV}(\sigma, N)$. The ring $A = \mathbb{C}[\sigma^\vee \cap M]$ as well as the modules $T^k_{X_\Delta}$ are M-graded, and the homogeneous pieces $T^k_{X_\Delta}(-R)$ with $R \in M$ may be described in terms of the polytope Δ: consider the complex

$$\begin{array}{ccccccccc}
0 \longrightarrow & C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & C^3 & \longrightarrow \cdots \\
& \| & & \| & & \| & & \| & \\
0 \longrightarrow & N_{\mathbb{C}} & \longrightarrow & \oplus_{f_0 \in \Delta} N_{\mathbb{C}}/\mathbb{C}\cdot f_0 & \longrightarrow & \oplus_{f_1 < \Delta} N_{\mathbb{C}}/\operatorname{span} f_1 & \longrightarrow & \cdots &
\end{array}$$

with $f_k < \Delta$ in the definition of C^{k+1} running through the k-dimensional faces of Δ; its cohomology is denoted by $D^k(\Delta) := H^k(C^\bullet)$. Then, in [2] we have shown that

Theorem 1. [2, (6.6)] *Assume that the two-dimensional faces of Δ are either squares or triangles with area 1 and 1/2, respectively, i.e. X_Δ is a conifold in codimension three. Then, if $R \in M$ is any degree, we have for $k \le 2$*

$$T^k_{X_\Delta}(-R) = \begin{cases} D^k\left(\Delta \cap [R=1]\right) & \text{if } R \le 1 \text{ on } \Delta \\ 0 & \text{otherwise}. \end{cases}$$

Moreover, the multiplication $x^s : T^k_X(-R) \to T^k_X(-(R-s))$ *with* $s \in \sigma^\vee \cap M$ *is induced from the restriction of* $D^k\left(\Delta \cap [R=1]\right)$ *to the face* $\Delta \cap [R-s=1] = \Delta \cap [R=1] \cap s^\perp$ *of* $\Delta \cap [R=1]$.

(2.2) The vector space $D^1(\Delta)$ or, to be more exact, a full-dimensional polyhedral cone in it, parametrizes the set of Minkowski summands of the polytope Δ: Each vertex of a Minkowski summand is considered a mutation of an original vertex $f_0 \in \Delta$ – hence it provides an element of the corresponding summand $N_\mathbb{C}/\mathbb{C}\cdot f_0$ of C^1. While it is possible to determine $D^1(\Delta)$, which leads to T^1_X, in many cases, we have to use a vanishing theorem for the D^2-invariant which is responsible for the obstructions. In [2] we have proved the following result:

Theorem 2. [2, (1.1) and (4.7)] *Let* Δ *be an n-dimensional, compact, convex polytope such that every three-dimensional face is a pyramid (with arbitrary base). If no vertex is contained in more than* $(n-3)$ *two-dimensional, non-triangular faces, then* $D^2(\Delta) = 0$.

3. Quiver polytopes

(3.1) Let Q be a connected quiver, i.e. an oriented graph. It consists of a set Q_0 of vertices (or "knots"), a set Q_1 of arrows, and two functions $t, h : Q_1 \to Q_0$ assigning to each arrow $\alpha \in Q_1$ its tail $t(\alpha)$ and head $h(\alpha)$. This gives rise to the incidence matrix Inc; it consists of $\#Q_0$ rows and $\#Q_1$ columns, and for $i \in Q_0, \alpha \in Q_1$ we have

$$\mathrm{Inc}_{i\alpha} := \begin{cases} +1 & \text{if } i = t(\alpha) \\ -1 & \text{if } i = h(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

The associated linear map $\pi : \mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_0}$, $\pi : [\alpha] \mapsto [t(\alpha)] - [h(\alpha)]$ provides an exact sequence

$$0 \to \mathbb{F} \xrightarrow{i} \mathbb{Z}^{Q_1} \xrightarrow{\pi} \mathbb{Z}^{Q_0} \xrightarrow{1} \mathbb{Z} \to 0$$

with some free abelian group $\mathbb{F}$ of rank $\#Q_1 - \#Q_0 + 1$. It is generated by the minimal, not necessarily oriented cycles in Q. Denote by $\mathbb{H} := \ker(\mathbb{Z}^{Q_0} \to \mathbb{Z})$ the so-called set of integral weights; it contains a canonical one defined as $\theta^c := \pi(\underline{1}) = \sum_{\alpha \in Q_1} \pi([\alpha])$.

Definition 3. To any weight $\theta \in \mathbb{H}_\mathbb{R} := \mathbb{H} \otimes_\mathbb{Z} \mathbb{R}$ we associate the so-called *flow polytope*

$$\nabla(Q, \theta) := \pi^{-1}(\theta) \cap \mathbb{R}^{Q_1}_{\ge 0}.$$

For non-connected quivers, the flow polytope is defined as the product of the flow polytopes associated to the connected components.

Proposition 4. *If $\theta \in \mathbb{H}$ is an integral weight, then $\nabla(Q, \theta)$ is a lattice polytope.*

Proof. (Communicated by G. M. Ziegler) The vertices of $\nabla(Q, \theta)$ may be obtained as the unique solutions of linear equations with a submatrix of Inc as coefficients and integers θ_p at the right hand side. On the other hand, square matrices having ± 1 as the only non-trivial entries with each of them occurring at most once in each column can only have determinant ± 1 or 0:

If every column contains both 1 and -1, then the rows add up to zero. Otherwise, there is a column having only one single non-trivial entry ± 1—and we use exactly this one to develop our determinant and end up with a smaller matrix. □

(3.2) There are two weights being of special interest:

(i) The *canonical weight* $\theta^c := \pi(\underline{1})$; we set $\nabla(Q) := \nabla(Q, \theta^c)$. The shifted polytope $\nabla(Q) - \underline{1} = \mathbb{F}_{\mathbb{R}} \cap \{r \in \mathbb{R}^{Q_1} | r_\alpha \geq -1\}$ contains 0 as an interior lattice point. This makes it possible to define the dual polytope as

$$\nabla(Q)^\vee := \left\{a \in \mathbb{F}^*_{\mathbb{R}} \,\middle|\, \langle a,\ \nabla(Q) - \underline{1}\rangle \geq -1\right\}.$$

$\nabla(Q)^\vee \subseteq \mathbb{F}^*_{\mathbb{R}}$ is the smallest polytope containing $0 \in \mathbb{F}^*$ and the so-called *quiver polytope*

$$\Delta(Q) := \operatorname{conv}\left\{a^\alpha := i^*([\alpha]) \in \mathbb{F}^* \mid \alpha \in Q_1\right\}.$$

In particular, both $\nabla(Q)^\vee$ and $\Delta(Q)$ are compact lattice polytopes.

(ii) The *zero weight* $\theta := 0$. Then, $\nabla(Q, 0) \subseteq \mathbb{F}$ is a polyhedral cone with apex; it is the dual cone of $\mathbb{R}_{\geq 0} \cdot \nabla(Q)^\vee = \mathbb{R}_{\geq 0} \cdot \Delta(Q)$. Moreover, $\nabla(Q, 0)$ occurs as the "tail cone" (cone of unbounded directions) $\nabla(Q, 0) = \nabla(Q, \theta)^\infty$ for every weight $\theta \in \mathbb{H}_{\mathbb{R}}$.

The quiver Q lacks oriented cycles if and only if

$$\nabla(Q, 0) = 0 \Longleftrightarrow \nabla(Q) \text{ is compact} \Longleftrightarrow 0 \in \operatorname{int} \nabla(Q)^\vee \Longleftrightarrow 0 \in \operatorname{int} \Delta(Q).$$

If this is the case, then $\Delta(Q) = \nabla(Q)^\vee$ is a reflexive polytope in the sense of Batyrev, cf. [3]. The corresponding affine toric Gorenstein singularity $X_{\Delta(Q)} := \mathbb{TV}(\sigma, N)$ with $N := \mathbb{F}^* \oplus \mathbb{Z}$ and $\sigma := \operatorname{cone}(\Delta(Q)) \subseteq N_{\mathbb{R}}$ will be our main subject of investigation; it equals the cone over the projective toric variety $\mathbb{P}_{\nabla(Q)} := \mathbb{TV}(\mathcal{N}(\nabla(Q)), \mathbb{F}^*)$ with $\mathcal{N}(\nabla(Q))$ denoting the normal fan of $\nabla(Q)$.

Example 5. Let Q be the quiver

$$\bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet$$

with the corresponding polytope

$$\Delta(Q) = \operatorname{conv}\{a^1, \ldots, a^6\} \subseteq \mathbb{R}^6 / \langle a^1 + a^2, a^3 + a^4, a^5 + a^6\rangle$$

being an octahedron with unit triangles as facets. In particular, since D^1 and D^2 of an octahedron is 0 and $\mathbb{C}^2$, respectively, Theorem 1 implies that $X_{\Delta(Q)}$ is rigid, but $T^2_X = T^2(-R^*)$ is two-dimensional. The singularity $X_{\Delta(Q)}$ is well known; its equations are the 2-minors of a general (2×3)-determinant. It illustrates the disappointing fact that T^2_X might contain more than just the obstructions, cf. Remark (6.5).

(3.3) By a subquiver $P \subseteq Q$ we mean a quiver P with $P_0 = Q_0$ and $P_1 \subseteq Q_1$. It is not assumed to be connected; in particular there might even occur isolated points $p \in P_0$. Fixing a weight $\theta \in \mathbb{H}_{\mathbb{R}}$, we will use the abbreviation $\theta(S) := \sum_{p\in S} \theta_p$ for subsets $S \subseteq Q_0$.

Definition 6. Let $\theta \in \mathbb{H}_{\mathbb{R}}$ be a weight of Q. A subquiver $P \subseteq Q$ is said to be

- *θ-(semi-) stable* (cf. [10]) if any non-trivial, proper subset $S \subset P_0$ that is closed under P-successors fulfills $\theta(S) < 0$ (or ≤ 0, respectively);
- *θ-polystable* if the connected components P^v of P, meant as connected quivers with a possibly smaller set $P_0^v \subseteq Q_0$, fulfill $\theta(P_0^v) = 0$ and, moreover, are θ-stable.

While these notions where defined in [10] to obtain decent moduli spaces of quiver representations, we will just use them to describe the faces of our quiver polytopes.

Lemma 7. (1) *"Stable" $\Longrightarrow$ "polystable" $\Longrightarrow$ "semistable".*
(2) *Stable quivers are always connected. Semistability of P implies $\theta(P_0^v) = 0$ for its connected components.*
(3) *The notions "stability" and "semistability" are closed under enlargement of the subquiver $P \subseteq Q$; "polystability" is not.*
(4) *A subquiver $P \subseteq Q$ is polystable if and only if $\nabla(P, \theta)$ contains a point with positive coordinates, i.e. if the set $\pi^{-1}(\theta) \cap \mathbb{R}^{P_1}_{>0}$ is non-empty.*
(5) *A subquiver $P \subseteq Q$ is semistable if and only if $\nabla(P, \theta) \neq \emptyset$.*
(6) *Every semistable subquiver $P \subseteq Q$ contains a (unique) maximal polystable subquiver $\bar{P} \subseteq P$.*

Proof. The first two parts are straightforward. Claim (3) follows from the easy fact that the larger $P \subseteq Q$, the more difficult is it for $S \subset Q_0$ to be closed under P-successors. However, the corresponding property fails for "polystability", since any enlargement of $P \subseteq Q$ may unify connected components.

To see that the condition in (4) suffices for polystability, let $S \subseteq Q_0$ be an arbitrary subset. We may use any $r \in \pi^{-1}(\theta)$ to calculate $\theta(S)$ as

$$\theta(S) = \sum_{S \stackrel{\alpha}{\to} (Q_0 \setminus S)} r_\alpha - \sum_{(Q_0 \setminus S) \stackrel{\alpha}{\to} S} r_\alpha .$$

Now, if $r \in \pi^{-1}(\theta) \cap \mathbb{R}^{P_1}$ and S is closed under P-successors, then the first sum is void. However, if S is not a union of connected components of P, then there must be at least one P-arrow connecting $Q_0 \setminus S$ and S, i.e. contributing to the second sum. In particular, if r has only positive coordinates, then $\theta(S) < 0$.

For proving the necessity of the condition, we may assume that $P = Q$ is θ-stable. If $\pi^{-1}(\theta) \cap \mathbb{R}^{Q_1}_{>0} = \emptyset$, then the vector $\theta = (\theta_p)_{p\in Q_0}$ may not be written as a positive linear combination of the columns of the incidence matrix Inc introduced in (3.1). Thus, duality in convex geometry provides the existence of a non-trivial vector $h \in \mathbb{R}^{Q_0}/(\mathbb{R} \cdot \underline{1})$ such that $\langle h, \mathrm{Inc}_{(\bullet,\alpha)} \rangle \geq 0$ for every arrow $\alpha \in Q_1$, but $\langle h, \theta \rangle \leq 0$. The first property means $h_{t(\alpha)} \geq h_{h(\alpha)}$. Hence, denoting by $c_1 < \cdots < c_k (k \geq 2)$ the values of h on Q_0 and choosing an arbitrary $c_0 < c_1$, the subsets

$$S_v := \{p \in Q_0 | h_p \leq c_v\} \subset Q_0 \quad (v = 0, \dots, k)$$

are closed under successors. In particular, by the stability of Q, we obtain $\theta(S_v) < 0$ for $v = 1, \dots, k-1$ and $\theta(S_0) = \theta(S_k) = 0$. This yields a contradiction via

$$0 < \sum_{v=1}^{k-1} (c_v - c_{v+1})\theta(S_v) = \sum_{v=1}^{k} c_v \cdot \theta(S_v \setminus S_{v-1}) = \sum_{p \in Q_0} h_p \theta_p \leq 0.$$

Finally, we have to show (5) and (6). First, if $P \subseteq Q$ is a subquiver with $\nabla(P, \theta) \neq \emptyset$, then we may choose an element $r \in \nabla(P, \theta)$ with maximal support $\bar{P}_1 := \{\alpha \in P_1 \mid r_\alpha > 0\}$. Hence, by (4), the corresponding subquiver $\bar{P}$ is polystable and, using (1) and (3), P must be semistable.

It remains to check that $\nabla(P, \theta) \neq \emptyset$ for semistable $P \subseteq Q$. We do this by copying the second part of the proof of (4) with minor changes: The stronger condition $\pi^{-1}(\theta) \cap \mathbb{R}^{Q_1}_{\geq 0} = \emptyset$ implies the existence of an h satisfying the strict inequality $\langle h, \theta \rangle < 0$. On the other hand, if Q is semistable, we have only $s_v \geq 0$. Nevertheless, one obtains the same contradiction. □

Corollary 8. *For a subquiver $P \subseteq Q$, we realize its flow polytope as the subset*

$$\nabla(P, \theta) = \{r \in \nabla(Q, \theta) \, | r_\alpha = 0 \text{ if } \alpha \notin P\}.$$

This provides an order preserving bijection between the poset of θ-polystable subquivers, on the one hand, and the face lattice of $\nabla(Q, \theta)$, on the other. In particular, since the dimension of $\nabla(P, \theta)$ is $(\#P_1 - \#Q_0 + \#(\text{components of } P))$, the θ-polystable trees yield the vertices of $\nabla(Q, \theta)$.

Proof. Every face of $\nabla(Q, \theta)$ is of the form $f = \{r \in \nabla(Q, \theta) | r_{\alpha^i} = 0, i = 1, \dots, k\}$ for some edges $\alpha^1, \dots, \alpha^k \in Q_1$. Assuming that the set $\{\alpha^1, \dots, \alpha^k\}$ is maximal for the given face f, we obtain P by $P_1 := Q_1 \setminus \{\alpha^1, \dots, \alpha^k\}$. □

(3.4) Every connected quiver Q is stable with respect to its *canonical weight* θ^c. In this situation, we had defined in (3.2) the polytopes $\Delta(Q) \subseteq \nabla(Q)^\vee$. In general, under the dualization $\nabla^\vee := \{a \mid \langle a, \nabla \rangle \geq -1\}$ of polytopes containing the origin, we obtain an anti-isomorphism of the posets

$$\begin{aligned} \Phi : \{\text{faces of } \nabla \text{ without } 0\} &\xrightarrow{\sim} \{\text{faces of } \nabla^\vee \text{ without } 0\} \\ f &\longmapsto \{a \in \nabla^\vee \mid \langle a, f \rangle = -1\}. \end{aligned}$$

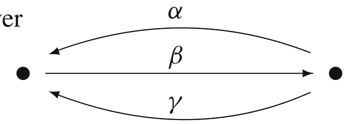
Note that faces containing 0 correspond to faces of the dual tail cone—this gives a kind of a continuation of Φ. If, e.g., $f \leq \nabla$ is as above and $[0, f] \leq \nabla$ denotes the smallest face containing 0 and f, then $\Phi([0, f])$ is the tail cone of $\Phi(f) \leq \nabla^\vee$.

Applying this to $\nabla := \nabla(Q) - \underline{1}$, we obtain for every θ^c-polystable subquiver $P \subseteq Q$ the face

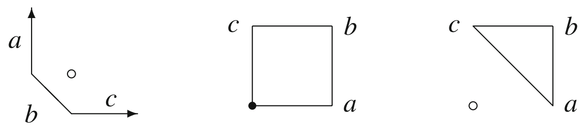
$$F(P, \Delta(Q)) := \Phi(\nabla(P, \theta^c_Q)) = \text{conv}\left\{a^\alpha \in \mathbb{F}^* \, | \alpha \notin P\right\}.$$

Since $F(P, \Delta(Q))$ does not contain 0, it is also a face of the quiver polytope $\Delta(Q)$.

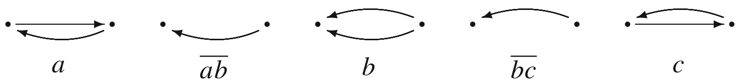
Example 9. With Q being the quiver



we obtain $\mathbb{F}^*_{\mathbb{R}} = \mathbb{R}^3/\langle a-b+c\rangle$. Using the basis $\{a, c\}$, we can draw the polyhedra $\nabla(Q) \subseteq \mathbb{F}_{\mathbb{R}}$ and $\nabla(Q)^\vee$, $\Delta(Q) \subseteq \mathbb{F}^*_{\mathbb{R}}$ as follows:



Here are the five proper θ^c-polystable subquivers giving rise to faces of them:



(3.5) Via contraction, we will construct new quivers $\Gamma_Q(P)$ which allow to consider the faces $F(P, \Delta(Q)) \leq \Delta(Q)$ as autonomous quiver polytopes. Note that even if Q has no oriented cycles, $\Gamma_Q(P)$ might have a lot of them.

Definition 10. For any subquiver $P \subseteq Q$ we define a quiver $\Gamma_Q(P)$. Its vertices $\Gamma_Q(P)_0$ are the connected components of P, and the arrows are defined as $\Gamma_Q(P)_1 := Q_1 \backslash P_1$. Every weight θ on Q induces a weight θ on $\Gamma_Q(P)$ with θ^c staying the canonical weight. If P was θ-polystable, then θ turns into the 0-weight on $\Gamma_Q(P)$.

Proposition 11. *Let $P < Q$ be a non-empty, θ^c-polystable subquiver. Then, the face $F(P, \Delta(Q)) < \Delta(Q)$ equals the quiver polytope $\Delta\left(\Gamma_Q(P)\right)$ and has dimension $(\#\Gamma_1 - \#\Gamma_0)$.*

Moreover, it is contained in a plane of $\mathbb{F}^*$ having lattice distance one from the origin.

Proof. Note that $\theta^c = 0$ in $\Gamma_Q(P)$. The original quiver Q and $\Gamma_Q(P)$ are related by the following commutative diagram where the vertical maps are surjective.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{Z}^{Q_1} \arrow[r] \arrow[d, "\text{forget}"] & \mathbb{Z}^{Q_0} \arrow[r] \arrow[d, "\text{sum}"] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{F}(\Gamma) \arrow[r] & \mathbb{Z}^{\Gamma_1} \arrow[r] & \mathbb{Z}^{\Gamma_0} \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Now, the first claim follows easily from the dual picture: $\mathbb{F}(\Gamma)^* \hookrightarrow \mathbb{F}^*$ is saturated, and $\mathbb{F}(\Gamma)^*$ is the image of $\mathbb{Z}^{\Gamma_1}$ under the surjection $\mathbb{Z}^{Q_1} \twoheadrightarrow \mathbb{F}^*$. The part concerning the height is a consequence of the fact that the faces of $\nabla(Q)^\vee$ are contained in affine hyperplanes $[r = -1] \subseteq \mathbb{F}^*$ for certain vertices $r \in \nabla(Q) - \underline{1}$. By Proposition 4, these r are contained in the lattice $\mathbb{F}$. □

Example 9 (continued). The $\Delta(Q)$-faces corresponding to the five θ^c-polystable subquivers are quiver polytopes arising from Γ consisting of one vertex and one or two loops.

4. Tightness

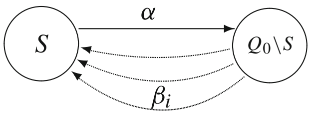
(4.1) To ensure that there is a one-to-one correspondence between arrows $\alpha \in Q_1$, on the one hand, and facets $\nabla(Q\backslash\alpha,\ \theta)$ of $\nabla(Q, \theta)$, on the other, we need the notion of tightness. In particular, if Q is θ^c-tight, then $a^\alpha \in \mathbb{F}^*$ will be the mutually distinct vertices of $\Delta(Q)$.

Definition 12. If $\theta \in \mathbb{H}$ is an integral weight, then the quiver Q is called *θ-tight* if for any $\alpha \in Q_1$ the subquiver $Q\backslash\alpha$ is θ-stable.

Lemma 13. (1) *Let $\theta \in \mathbb{H}$ such that Q is θ-stable. By contraction of certain arrows in Q, the weight θ becomes tight without changing the polytope $\nabla(Q, \theta)$. Moreover, the canonical weight θ^c may be tightened in such a way that not only the polytope $\nabla(Q, \theta^c)$, but also $\nabla(Q, \theta)$ remains untouched.*
(2) *Assume that Q is θ-tight. If $P \subseteq Q$ is a θ-polystable subquiver, then, $\Gamma_Q(P)$ becomes 0-tight.*
(3) *Let Γ be a 0-tight quiver with $\#\Gamma_0 \geq 2$. Then, not counting the loops, every knot of Γ has at least two in- and outgoing arrows, respectively. In particular,*

$$\#\Gamma_1 \geq 2\#\Gamma_0 + \#(\text{loops of } \Gamma).$$

Proof. (1) If $Q\backslash\{\alpha\}$ is not θ-stable, then there exists a subset $S \subset Q_0$ that is, up to α, closed under successors and satisfies $\theta(S) \geq 0$. Let $\beta_1, \ldots, \beta_l$ be the arrows pointing from $Q_0\backslash S$ to S; since Q is θ-stable, α leads in the opposite direction.



If $r \in \pi^{-1}(\theta)$, then $r_\alpha = \sum_i r_{\beta_i} + \theta(S) \geq \sum_i r_{\beta_i}$. Hence, the relations $r_{\beta_i} \geq 0$ together with the $\pi^{-1}(\theta)$-equations force that $r_\alpha \geq 0$. In particular, we may contract α without changing the polytope $\nabla(Q, \theta)$ (including its lattice structure).

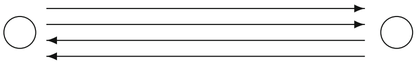
Tightening θ^c, we obtain $1 - l = \theta^c(S) \geq 0$. In case of $l = 0$, the θ-stability of Q implies that $\theta(S) > 0$. In particular, contracting α does not change neither $\nabla(Q, \theta^c)$, nor $\nabla(Q, \theta)$. If $l = 1$, then the situation is symmetric with $\theta^c(S) = 0$. Depending on whether $\theta(S) \geq 0$ or $\theta(S) \leq 0$, we should contract α or β_1, respectively.

(2) Let $\alpha \in \Gamma_1 = Q_1\backslash\Delta_1$. Then, the connectivity of $Q\backslash\alpha$ implies the connectivity of $\Gamma\backslash\alpha$. Moreover, projecting any positive $r \in \nabla(Q\backslash\alpha, \theta)$ along the forgetful map $\mathbb{Z}^{Q_1} \to \mathbb{Z}^{\Gamma_1}$ (see the diagram of the proof of Proposition 11), provides an $\overline{r} \in \nabla(\Gamma\backslash\alpha, 0)$ with positive entries.

(3) For every $\alpha \in \Gamma_1$, non-trivial subsets $S \subseteq \Gamma_0$ cannot be closed under $(\Gamma\backslash\alpha)$-successors. Hence, there is always a $\beta \in \Gamma_1\backslash\alpha$ leaving S. Now, the claim follows from applying this fact to the cases $\#S = 1$ or $\#(\Gamma_0\backslash S) = 1$. □

(4.2) If Q has no oriented cycles, then $\Delta(Q) = \nabla(Q)^\vee$, and Proposition 11 yields all its faces—they equal $\Delta(\Gamma)$ with $\Gamma = \Gamma_Q(P)$ for θ^c-polystable subquivers $P \leq Q$. Moreover, Q and hence Γ maybe assumed to be θ^c-tight. In particular, $\#\Gamma_1 \geq 2\#\Gamma_0 + \#(\text{loops of } \Gamma)$.

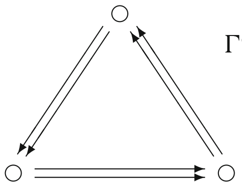
Using this, we are now classifying all possible faces of those polytopes $\Delta(Q)$ up to dimension three. Note that $\theta^c = 0$ in Γ.
Dimension one: Γ consists of one vertex with two loops. The corresponding polytope $\Delta(\Gamma)$ is a lattice interval of length one.
Dimension two: The case $\#\Gamma_0 = 1$ yields the triple loop with $\Delta(\Gamma)$ being the standard triangle. On the other hand, there is only one quiver with $\theta^c(\Gamma) \equiv 0$ that consists of two vertices, four arrows, but no loops:



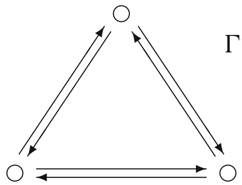
$\Gamma^{\text{dbl}}(2) = \Gamma^{\text{opp}}(2)$

The corresponding polytope $\Delta(\Gamma)$ is the unit square.
Dimension three: The case $\#\Gamma_0 \leq 2$ yields the quivers of the previous list with one additional loop. Adding a loop to Γ corresponds to taking the pyramid of height 1 over the corresponding polytope $\Delta(\Gamma)$. In particular, we obtain the unit tetrahedron and the pyramid over the unit square.

On the other hand, there are two different quivers involving three vertices and six edges:



$\Gamma^{\text{dbl}}(3)$



$\Gamma^{\text{opp}}(3)$

The first quiver provides an octahedron. However, compared with the quiver polytope presented in Example (3.2), the central point does no longer belong to the lattice. The other quiver provides the prism of height 1 over the unit triangle.

Corollary 14. *Let Q be a quiver without oriented cycles. Then, $\Delta(Q)$ and its faces satisfy the assumptions of Theorem 1: The two-dimensional faces are either squares or triangles with area* 1 *and* 1/2*, respectively. Thus, $X_{\Delta(Q)}$ is a conifold in codimension three, and the vector spaces T^i_X may be obtained by calculating the corresponding D-invariants of the $\Delta(Q)$-faces.*

(4.3) Let Q be a quiver without oriented cycles. We conclude this chapter with determining all proper faces of $\Delta(Q)$ having a non-trivial D^1 (cf. (2.1)), i.e. admitting a non-trivial splitting into Minkowski summands.

Example 15. If $\pi \in S_k$ is a permutation, then we denote by $\Gamma(k, \pi)$ the quiver with

- vertex set $\Gamma(k, \pi)_0 = \mathbb{Z}/k\mathbb{Z}$ and
- arrows $\beta_p, \gamma_p \in \Gamma(k, \pi)$ defined as $\beta_p : p \to (p+1)$ and $\gamma_p : p \to \pi(p)$ for $p = 1, \ldots, k$.

The permutations $\pi^{\text{dbl}}(p) := p+1$ and $\pi^{\text{opp}}(p) := p-1$ are of special interest; the quivers $\Gamma^{\text{dbl}}(k) := \Gamma(k, \pi^{\text{dbl}})$ and $\Gamma^{\text{opp}}(k) := \Gamma(k, \pi^{\text{opp}})$ are double n-gons as shown in (4.2) for $k = 2$ and $k = 3$. The corresponding polytopes are

$$\Delta\left(\Gamma^{\text{dbl}}(k)\right) = [\text{crosspolytope of dimension } k] \quad \text{and} \quad \Delta\left(\Gamma^{\text{opp}}(k)\right) = \Delta^{k-1} \times [0, 1].$$

In particular, while $D^1\,(\Delta(\Gamma^{\mathrm{dbl}}(k))) = 0$ for $k \geq 3$, we have $\dim D^1\,(\Delta(\Gamma^{\mathrm{opp}}(k))) = 1$ with the obvious Minkowski decomposition.

Lemma 16. *Let Γ be a quiver which is tight with respect to $\theta^c = 0$. Assume that $b \subseteq \Gamma_1$ is a simple cycle, i.e. not touching vertices twice.*

(1) *Contracting b, the resulting quiver $\bar{\Gamma} := \Gamma/b$ is still tight with respect to $\theta^c = 0$. Moreover, $\Delta(\bar{\Gamma})$ is a face of $\Delta(\Gamma)$ inducing the restriction map $p : D^1\,(\Delta(\Gamma)) \to D^1\,\big(\Delta(\bar{\Gamma})\big)$.*
(2) *Unless $\Gamma = \Gamma(k, \pi)$ and b is a cycle of length k, the map p is injective.*

Proof. Let $b = \{\alpha^1, \ldots, \alpha^k\}$ and denote by a^i the image of $[\alpha^i]$ in $\mathbb{F}^*$. The relations among the vertices of $\Delta(\Gamma)$ which are induced from $\bar{\Gamma}$-knots are exactly the relations among the vertices of $\Delta(\bar{\Gamma})$. On the other hand, the knots being touched by b express $(a^{i+1} - a^i)$ as an element of the vector space associated to the affine space $\mathbb{A}$ spanned by $\Delta(\bar{\Gamma})$. In particular, $\Delta(\bar{\Gamma})$ is a face of $\Delta(\Gamma)$ and, moreover, the remaining vertices $a^1, \ldots, a^k$ are contained in an affine plane being parallel to $\mathbb{A}$; they form their own face $B := \mathrm{conv}\{a^1, \ldots, a^k\}$.

Let $\underline{t} \in D^1\,(\Delta(\Gamma))$—here we think of it as a tuple of dilatation factors for every compact edge of $\Delta(\Gamma)$: The factors arise after applying the differential $d^1 : C^1 \to C^2$ from (2.1); since it yields 0, all the components of the image must be contained in the subspaces span f_1.

Since all vertical edges connecting $\Delta(\bar{\Gamma})$ and B have the same dilatation factor, we may assume these factors to be zero. Now, if $p(\underline{t}) = 0$, then the dilatation factors inside $\Delta(\bar{\Gamma})$ are also mutually equal; it remains to show that they vanish. If not, then we get a map

$$\{a^1, \ldots, a^k\} \longrightarrow \{\text{vertices of } \Delta(\bar{\Gamma})\}$$

assigning a^i the only vertex $a \in \Delta(\bar{\Gamma})$ such that $\overline{aa^i}$ is an edge of $\Delta(\Gamma)$. Since this map is obviously surjective, we obtain

$$\#\bar{\Gamma}_1 \leq k = \#b.$$

Hence, Γ equals b with an additional arrow leaving and reaching each knot. □

Proposition 17. *Let Q be a θ^c-tight quiver without oriented cycles. Then, the only proper, k-dimensional faces $F(P, \Delta(Q))$ of $\Delta(Q)$ having a non-trivial D^1 are those with $\Gamma_Q(P) \cong \Gamma^{opp}(k)$.*

Proof. Let $\Gamma := \Gamma_Q(P)$. In the proof of the previous lemma we have seen that the existence of a loop, i.e. of a cycle of length 1, implies that $\Delta(\Gamma)$ is a pyramid over $\Delta(\bar{\Gamma})$. In particular, it has a trivial D^1.

On the other hand, via a successive contraction of simple cycles of length $k_i \geq 2$, we can produce a sequence of quivers, beginning with Γ, such that

- we avoid the contraction of k_i-cycles in quivers isomorphic to $\Gamma(k_i, \pi)$, and
- we end with either the existence of loops or with a quiver isomorphic to some $\Gamma^{\mathrm{dbl}}(m)$. The latter leads to a non-trivial D^1 only for $m = 2$.

By Lemma 16(2), this sequence represents $D^1(\Delta(\Gamma))$ as a subset of either 0 or $D^1(\Delta(\Gamma^{\text{dbl}}(2))) = \mathbb{C}$. On the other hand, if the contraction of a simple k_i-cycle leads from Γ^i to Γ^{i+1}, then

$$\#(\Gamma_0^{i+1}) = \#(\Gamma_0^i) - (k_i - 1) \quad \text{and} \quad \#(\Gamma_1^{i+1}) = \#(\Gamma_1^i) - k_i .$$

In particular, relations like $\#(\Gamma_1^i) \geq 2\,\#(\Gamma_0^i)$ or $\#(\Gamma_1^i) > 2\#(\Gamma_0^i)$ will be inherited from Γ^i to Γ^{i+1}. If $k_i \geq 3$, then the weak inequality for Γ^i does even imply the strict one for Γ^{i+1}. Thus, if our sequence ends with $\Gamma^{\text{dbl}}(2)$, only 2-cycles have been contracted successively from Γ. This enforces that $\Gamma \cong \Gamma^{\text{opp}}(k)$. □

5. Flag quivers

(5.1) First, we will describe the classes of Weil and Cartier divisors on the projective variety $\mathbb{P}_{\nabla(Q,\theta)}$ provided by a general quiver Q with weight $\theta \in \mathbb{H}$. Assume, w.l.o.g., that Q is θ-tight. We introduce the notation

$$\mathcal{T}^\theta(Q) := \{\theta\text{-polystable trees } T < Q\}.$$

The tightness of Q implies that $\bigcap \mathcal{T}^\theta(Q) = \emptyset$. For $T \in \mathcal{T}^\theta(Q)$, or more general for any θ-polystable subquiver $P < Q$, we may define

$$\mathbb{H}(P) := \mathbb{H}\left(\Gamma_Q(P)\right) = \ker\left(\mathbb{Z}^{P\text{-comp}} \xrightarrow{\deg} \mathbb{Z}\right)$$

with $\mathbb{H}$ and $\Gamma_Q(P)$ as mentioned in (3.1) and Definition 10, respectively.

Proposition 18. *The class group* $\operatorname{DivCl}(\mathbb{P}_{\nabla(Q,\theta)})$ *of Weil divisors equals* $\mathbb{H}$.

Proof. Equivariant Weil divisors correspond to maps $\mathcal{N}^{(1)} \to \mathbb{Z}$ with $\mathcal{N}$ denoting the inner normal fan $\mathcal{N}(\nabla(Q,\theta))$ of $\nabla(Q,\theta) \subseteq \mathbb{F}_{\mathbb{R}}$. Since the elements of $\mathcal{N}^{(1)}$ encode the facets of $\nabla(Q,\theta)$, i.e. the θ-polystable subquivers $Q\backslash\alpha$, the equivariant Weil divisors may be written as elements of $\mathbb{Z}^{Q_1}$.

On the other hand, as being well known in the theory of toric varieties, the principal divisors among the equivariant ones are given by $\mathbb{F} \subseteq \mathbb{Z}^{Q_1}$. Hence, the claim follows from

$$\mathbb{Z}^{Q_1}/\mathbb{F} = \operatorname{im}\left(\mathbb{Z}^{Q_1} \to \mathbb{Z}^{Q_0}\right) = \ker\left(\mathbb{Z}^{Q_0} \xrightarrow{\deg} \mathbb{Z}\right) = \mathbb{H}\,.$$

□

(5.2) Using the fact that divisors on a toric variety are locally principal if and only if they are principal on the equivariant affine charts, we obtain a description of $\operatorname{Pic}(\mathbb{P}_{\nabla(Q,\theta)})$ as well.

Proposition 19. *The Picard group* $\operatorname{Pic}(\mathbb{P}_{\nabla(Q,\theta)})$ *of the projective toric variety* $\mathbb{P}_{\nabla(Q,\theta)}$ *equals*

$$\operatorname{Pic}(\mathbb{P}_{\nabla(Q,\theta)}) = \ker\left(\mathbb{H} \to \oplus_{T\in\mathcal{T}^\theta}\mathbb{H}(T)\right).$$

Proof. An element $g \in \mathbb{Z}^{Q_1}$ represents a Cartier divisor if and only if g induces principal divisors on the affine charts $\mathbb{TV}(\delta(T))$ with $\delta(T) := \langle a^\alpha \in \mathbb{F}^* | \alpha \notin T \rangle \in \mathcal{N}(\nabla(Q,\theta))$ for every θ-polystable tree $T \in \mathcal{T}^\theta$. This condition is equivalent to $g \in \mathbb{Z}^{T_1} + \mathbb{F} \subseteq \mathbb{Z}^{Q_1}$. Adapting the commutative diagram from the proof of Proposition 11, we obtain

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathbb{Z}^{T_1} \cap \mathbb{F} \arrow[r] \arrow[d] & \mathbb{Z}^{T_1} \arrow[d] & & \\
0 \arrow[r] & \mathbb{F} \arrow[r] \arrow[d] & \mathbb{Z}^{Q_1} \arrow[r] \arrow[d, "\text{forget}"] & \mathbb{H} \arrow[r] \arrow[d, "\text{sum}"] & 0 \\
0 \arrow[r] & \mathbb{F}(\Gamma) \arrow[r] \arrow[d] & \mathbb{Z}^{\Gamma_1} \arrow[r] \arrow[d] & \mathbb{H}(T) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It implies that the membership $g \in \mathbb{Z}^{T_1} + \mathbb{F}$ translates into the fact that the class $\bar{g} \in \mathbb{H}$ maps to $0 \in \mathbb{H}(T)$. □

Corollary 20. $\operatorname{Pic}(\mathbb{P}_{\nabla(Q,\theta)}) = \big\{\theta' \in \mathbb{H} \,\big|\, \theta'(S) = 0 \text{ for } S \subseteq Q_0 \text{ with } \theta(S) = 0$ $\text{and } Q_{|S} \text{ and } Q_{|(Q_0\setminus S)} \text{ being } \theta\text{-semistable}\big\}$.
In particular, a necessary, but only on $\theta \in \mathbb{H}$ *depending condition for* $\operatorname{Pic}(\mathbb{P}_{\nabla(Q,\theta)}) \cong \mathbb{Z}$ *is that* $\big\{\theta' \in \mathbb{H}_\mathbb{R} \mid \theta'(S) = 0 \text{ for } S \subseteq Q_0 \text{ with } \theta(S) = 0\big\} = \mathbb{R} \cdot \theta$.

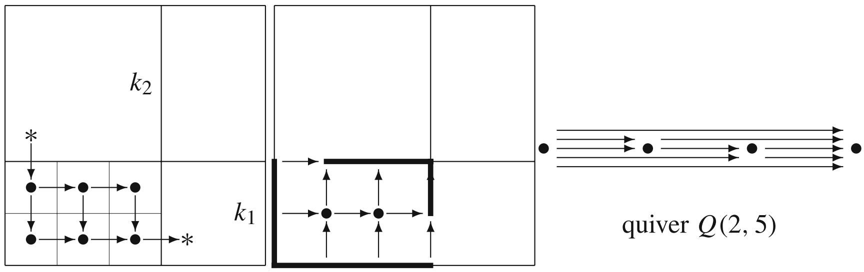
(5.3) Now, we turn to the main point of this section—the introduction of the so-called flag quivers. They will allow an easy description of their Picard group as well as, in Sect. 6, of their deformation theory.

Definition 21. A quiver Q without oriented cycles is called a *flag quiver* if

(i) there is exactly one source $p^0 \in Q_0$ with $m := \theta^c(p^0)$,
(ii) there are sinks $p^1, \dots, p^l$ with $m_i := -\theta^c(p^i) \geq 2$, and
(iii) the canonical weight vanishes on the remaining knots, i.e. on $Q_0 \setminus \{p^0, \dots, p^l\}$. In particular, we have $m = \sum_{i=1}^l m_i$.

Remark. The condition "$m_i \geq 2$" may be explained as follows: If $m_i = 1$, then Q cannot be tight, cf. Lemma 22. Moreover, tightening would mean to contract the arrow pointing to p^i, hence to create a non-sink with negative weight.

(5.4) The name "flag quiver" arises from the following example from [7]: For positive integers $k_1, \dots, k_{l+1}$, we set $n_i := \sum_{v=1}^i k_v$ and $n := n_{l+1}$. Then, a quiver $Q(n_1, \dots, n_l, n)$ may be defined via an $(n \times n)$-ladder-box containing the $(k_i \times k_i)$-squares on the main diagonal. As depicted in the middle figure below, Q_0 consists of the interior lattice points in the region below the small squares and of the closures of the $(l+1)$ connected components of the part of the boundary of this region that avoids the south west corners of the $(k_i \times k_i)$-squares. As arrows we take all possibilities pointing eastbound and northbound.



In [7], the authors have originally considered $Q^*(n_1, \ldots, n_l, n)$ as depicted in the left figure above: It is a kind of a dual quiver; its ordinary vertices correspond to the boxes instead of the lattice points, and there are additional, exceptional, vertices called "$*$" sitting in the south west corners of the $(k_i \times k_i)$-squares. Nevertheless, it was shown that the corresponding $X_{\Delta(Q)}$ equals the cone over a projective toric variety being a degeneration of the flag manifold $\mathrm{Flag}(n_1, \ldots, n_l, n)$. The polytopes assigned to the quiver $Q(n_1, \ldots, n_l, n)$ are called $\nabla(n_1, \ldots, n_l, n)$ and $\Delta(n_1, \ldots, n_l, n)$, respectively.

(5.5) The polystability of subquivers has an easy meaning in the special case of flag quivers:

Lemma 22. *Let Q be a flag quiver. $P \subseteq Q$ is θ^c-polystable iff it is a union of paths leading from p^0 to* every *sink p^i $(i = 1, \ldots, l)$. Moreover, Q is tight if and only if there are no verices with only one in- and outgoing arrow, respectively. In particular, tightening preserves the property of being a flag quiver.*

Proof. Both the criterion for polystability and the necessity of the tightness condition for Q are clear.

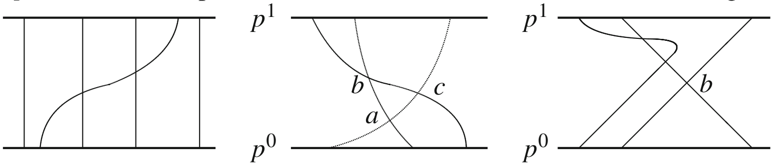
On the other hand, assume that Q satisfies this condition and let $\alpha \in Q_1$ be an arbitrary arrow. We may choose paths r^v leading from p^0 to p^v, but avoiding α. Moreover, for any $\beta \neq \alpha$ there is a special path $s(\beta)$ which, additionally, touches β; let $p^{\mu(\beta)}$ be the sink reached by $s(\beta)$. Then, with

$$R(\beta) := s(\beta) \cup \left(\bigcup\nolimits_{v \neq \mu(\beta)} r^v \right)$$

we have obtained a union of paths encoding a polystable subquiver containing β, but not α. In particular, $\bigcup_{\beta \neq \alpha} R(\beta)$ shows the polystability of the quiver $P = Q \setminus \alpha$. □

Tight flag quivers may be visualized as a so-called *fence*, i.e. as a system of mutually intersecting ropes leading from the l different ceilings $p^1, \ldots, p^l$ to the only base p^0. The intersections correspond to the knots $b \in Q_0 \setminus \{p^0, \ldots, p^l\}$. If, moreover, the quiver is a plane one, then, by Corollary 8, the dimension of the corresponding polytopes $\nabla(Q)$ and $\Delta(Q)$ may be read off the plane fence as the number of compact regions.

Example 23. Here, we present three fences of dimensions six, five, and again five.



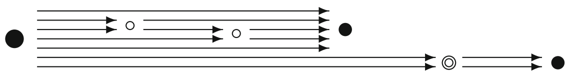
$Q(2, 5)$ Q^1 Q^2

(5.6) Assume that Q is a tight flag quiver. Denoting by $B \subseteq Q_0 \backslash \{p^0, \ldots, p^l\}$ the set of *blocking knots*, i.e. those that are not avoidable in a set of paths leading from p^0 to each of the ends $p^1, \ldots, p^l$, respectively, the Picard number of $\mathbb{P}_{\nabla(Q)}$ will be $\#(B) + l$. More precisely, we obtain

Proposition 24. $\mathrm{Pic}(\mathbb{P}_{\nabla(Q)}) = \ker \left(\mathbb{Z}^{\{p^0,\ldots,p^l\} \cup B} \xrightarrow{\deg} \mathbb{Z}\right)$.

Proof. The θ^c-polystable subquivers are the unions of paths leading from p^0 to $\{p^1, \ldots, p^l\}$. In particular, if $\theta' \in \mathrm{Pic}\,(\mathbb{P}_{\nabla(Q)})$, then $\theta'(b) = 0$ for vertices $b \notin \{p^0, \ldots, p^l\} \cup B$. On the other hand, since for each S as in Corollary (5.2) either S or $Q_0 \backslash S$ contains $\{p^0, \ldots, p^l\} \cup B$, this remains the only condition. □

Example 25.



$\nabla(2, 5) \times \mathbb{P}^1 \times \mathbb{P}^1$ with Picard number 3

Knots $x \in B$ give rise to a decomposition of Q into a join of smaller quivers, meaning that $\mathbb{P}_{\nabla(Q)}$ splits into a product of projective varieties.

6. Deformation theory of flag quivers

(6.1) Let Q be a tight flag quiver. From Theorem 1 and Corollary 14 we know that the module T^1_X of infinitesimal deformations of $X_{\Delta(Q)} = \mathrm{Cone}(\mathbb{P}_{\nabla(Q)})$ is built from the spaces of Minkowski summands $D^1(F)$ of the faces $F \le \Delta(Q)$. These faces look like $F(P, \Delta(Q)) = \Delta(\Gamma_Q(P))$ for θ^c-polystable subquivers $P \le Q$, and, by Proposition 17, $\Gamma_Q(P)$ must be isomorphic to $\Gamma^{\mathrm{opp}}(k)$ to yield a non-trivial D^1. On the other hand, in the special case of flag quivers, Lemma 22 provides an explicit description of the θ^c-polystable subquivers at all. Combining all this information, we will get a complete description of T^1_X.

Definition 26. A knot $b \in Q_0 \backslash \{p^0, \ldots, p^l\}$ in a tight flag quiver is called *simple* if

- b is of valence four, i.e. supporting exactly two pairs of in- and outgoing arrows, respectively, and
- both pairs neither have, besides b, a common tail or head of valence four in the set $Q_0 \backslash \{p^0, \ldots, p^l\}$, nor a common head p^i with $m_i = 2$, nor the common tail p^0 with $m = 2$.

Visualizing Q as a fence, then simple knots correspond to the simple crossings of two ropes that are not adjacent to a further simple crossing of the same two ropes.

Example 23 (continued). In the first two quivers $Q(2,5)$ and Q^1 of Example 23, every knot of $Q\backslash\{p^0, p^1\}$ is simple. On the other hand, in Q^2, only b shares this property. The remaining two knots provide for each other the reason to violate the condition described in the previous definition.

Proposition 27. *Let Q be a tight flag quiver with* $\dim \Delta(Q) \geq 3$. *Then, the only faces $F \leq \Delta(Q)$ admitting a non-trivial Minkowsi decomposition are the two-dimensional squares $F(Q\backslash b)$ with b being a simple knot.*
($Q\backslash b \subseteq Q$ denotes the subquiver obtained by erasing the four arrows containing b).

Proof. First, we consider the *proper* faces of $\Delta(Q)$. Lemma 22 tells us that θ^c-polystable subquivers P consist of one major component and a bunch of isolated knots from $Q_0\backslash\{p^0, \ldots, p^l\}$. On the other hand, the only quivers providing a non-trivial D^1 are $\Gamma^{\text{opp}}(k)$. If $\Gamma_Q(P) = \Gamma^{\text{opp}}(k)$ with $k \geq 3$, then Q would have to contain oriented cycles. Thus, $P = Q\backslash\{b\}$ for some knot b.

Denote by α^1, α^2 and β^1, β^2 the pairs of in- and outgoing b-arrows, respectively. Assuming that, for instance, α^1 and α^2 had a common tail $c \in Q_0\backslash\{p^0, \ldots, p^l\}$ of valence four, then the two arrows having c as common head could not occur in paths avoiding b and leading from p^0 to $\{p^1, \ldots, p^l\}$. In particular, $P = Q\backslash\{b\}$ would not be stable. The reversed implication may be shown along the lines of the proof of Lemma 22.

It remains to deal with the polytope $\Delta(Q)$ itself. If $\dim \Delta(Q) \geq 4$, then we have just shown that every facet is Minkowski prime; this implies the same property for $\Delta(Q)$, too. The three-dimensional case can be easily solved by a complete classification. □

Corollary 28. *For $X_{\Delta(Q)}$, the non-trivial $T^1_X(-R)$ are one-dimensional, and they arise from degrees $R = R(b)$ such that $R(b) \leq 1$ on $\Delta(Q)$ and $R(b) = 1$ exactly on $F(Q\backslash b)$ with b being a simple knot.*

(6.2) The precise description of T^1_X given in Corollary 28 may be supplemented by the following, more structural claim.

Theorem 29. *Let $X_{\Delta(Q)}$ be the toric Gorenstein singularity assigned to the quiver polytope of a flag quiver Q. Then, the simple knots $b \in Q_0\backslash\{p^0, \ldots, p^l\}$ parametrize the three-codimensional A_1-strata $Z(b) \xhookrightarrow{i(b)} X_{\Delta(Q)}$, and the module of the infinitesimal deformations of $X_{\Delta(Q)}$ equals*

$$T^1_X = \oplus_b\, i(b)_*\, \omega_{Z(b)} \otimes \omega_X^{-1}$$

with $\omega_{\ldots}$ denoting the canonical sheaves on $Z(b)$ and X.

Proof. We know from Corollary 14 that the three-codimensional singularities in $X_{\Delta(Q)}$ correspond to the two-dimensional, non-triangular faces of $\Delta(Q)$ which are squares. On the other hand, Proposition 27 establishes their relation to the simple knots b. For any of it we may define

$$T(b) := \oplus_{R(b)} T^1\,(-R(b)) = D^1\,(F(Q\backslash b))^{\#\{R(b)'\mathrm{s}\}}$$

meaning to sum over all $R(-b)$ in the sense of Corollary 28. Thus, the whole T^1_X splits, as a $\mathbb{C}$-vector space, into $T^1_X = \oplus_b T(b)$.

The module structure of T^1_X has been explained in Theorem 1: The dual cone of $\sigma = \mathrm{cone}(\Delta(Q)) \subseteq N_{\mathbb{R}}$ is $\sigma^\vee = \mathrm{cone}(\nabla(Q)) \subseteq M_{\mathbb{R}}$ with $M := \mathrm{Hom}(N, \mathbb{Z}) = \mathbb{F} \oplus \mathbb{Z}$, cf. (3.2). Whenever $s \in \sigma^\vee \cap M$ vanishes on $F(Q\backslash b)$, i.e. if $s \in \mathrm{cone}\,\nabla(Q\backslash b, \theta^c_Q) \subseteq \mathrm{cone}\,\nabla(Q)$, then the multiplication $x^s : T^1\,(-R(b)) \to T^1\,(-R(b)+s)$ is the identity map when both sides are identified with the one-dimensional $D^1\,(F(Q\backslash b))$. If s does not vanish on $F(Q\backslash b)$, then the multiplication is zero.

Hence, the splitting of T^1 respects the module structure. Moreover, on the summands $T(b)$, this structure factors through the surjection $\mathbb{C}[\sigma^\vee \cap M] \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap F(Q\backslash b)^\perp \cap M] = \mathbb{C}[\mathrm{cone}\,\nabla(Q\backslash b, \theta^c_Q) \cap M]$ with

$$T(b) = \left(\oplus_{R \in \mathrm{int\,cone}\,\nabla(Q\backslash b,\, \theta^c_Q)}\, \mathbb{C} \cdot x^{R-\underline{1}}\right) \otimes_{\mathbb{C}} D^1\,(F(Q\backslash b))\,.$$

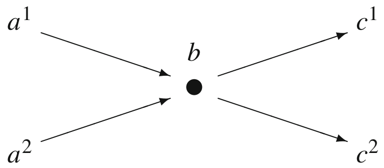
On the other hand, the semigroup algebra $\mathbb{C}[\mathrm{cone}\,\nabla(Q\backslash b, \theta^c_Q) \cap M]$ equals the coordinate ring of the stratum $Z(b)$, and it is a general fact for affine toric varieties $\mathbb{TV}(\tau) = \mathrm{Spec}\,\mathbb{C}[\tau^\vee \cap M]$ that $\oplus_{R \in \mathrm{int}\,\tau^\vee} \mathbb{C} \cdot x^R$ equals the canonical module $\omega_{\mathbb{TV}(\tau)}$. □

(6.3) Eventually, we would rather like to study the deformations of the projective toric variety $\mathbb{P}_{\nabla(Q)}$ instead of that of its cone $X_{\Delta(Q)}$. This just means to focus on those multidegrees R with height or $\mathbb{Z}$-degree 0, i.e. on $R \in \mathbb{F} \times \{0\} \subseteq \mathbb{F} \times \mathbb{Z} = M$. To use Corollary 28 for describing the entire homogeneous piece $T^1_X(0)$, it is helpful to realize M as a subspace of $\mathbb{Z}^{Q_1}$. This is done by the isomorphism

$$\begin{aligned} \mathbb{F} \oplus \mathbb{Z} \xrightarrow{\sim} \pi^{-1}(\mathbb{Z} \cdot \theta^c), \qquad (r, g) &\longmapsto r + g\underline{1} \\ (\nabla - \underline{1}, 1) &\longmapsto \nabla. \end{aligned}$$

Under this map, the value of $R = (r, g) \in M$ on the vertex $(a^\alpha, 1)$ of $(\nabla - \underline{1}, 1)$ equals the $[\alpha]$-coordinate of $R = r + g\underline{1} \in \mathbb{Z}^{Q_1}$. In particular, multidegrees R of height 0 are exactly those coming from $\pi^{-1}(0 \cdot \theta^c) = \mathbb{F} \subseteq \mathbb{Z}^{Q_1}$.

Definition 30. Let b be a simple knot and denote by a^1, a^2 the tails of the two arrows α^1, α^2 with head b, respectively; c^1, c^2 are defined in a similar manner on the outgoing arrows β^1, β^2.



An element $R \in \mathbb{Z}^{Q_1}_{\geq 0}$ is called *multipath* from $\{a^1, a^2\}$ to $\{c^1, c^2\}$ if $\pi(R) = [a^1]+[a^2]-[c^1]-[c^2]$. The standard example is $R_b := [\alpha^1]+[\alpha^2]+[\beta^1]+[\beta^2]$ through b.

Theorem 31. *Running through the simple knots $b \in Q_0 \backslash \{p^0, \ldots, p^l\}$, the part $T^1_X(0)$ splits into $T^1_X(0) = \oplus_b T_0(b)$, and the dimension of each $T_0(b)$ equals the number of multipaths leading from $\{a^1, a^2\}$ to $\{c^1, c^2\}$, but avoiding b.*

Proof. Corollary 28 characterizes the T^1-carrying multidegrees $R(b) \in \mathbb{Z}^{Q_1}$ assigned to the knot b by the properties

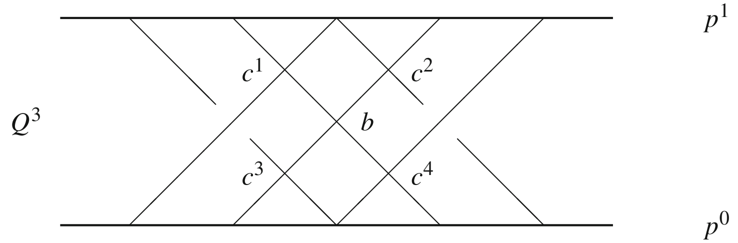
- $R(b)_\alpha = 1$ for α being one of the four arrows touching b and
- $R(b)_\alpha \leq 0$ for the remaining arrows $\alpha \in (Q\backslash b)_1$.

On the other hand, the condition of having height 0 means $R(b) \in \mathbb{F}$, i.e. that $R(b)$ encodes an cycle inside Q. Hence, the negative multidegrees $-R(b)$ represent cycles using each of the four b-arrows exactly once and in the wrong direction, but respecting the orientation of the remaining arrows in Q. With other words, $R_b - R(b)$ represents a multipath from $\{a^1, a^2\}$ to $\{c^1, c^2\}$ avoiding b. □

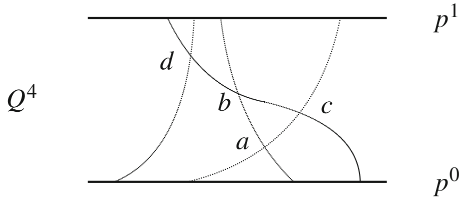
Example 23 (continued). While, in the quiver Q^1 of (5.5), the vertices a and b give rise to unique multidegrees $R(a)$ and $R(b)$ of height 0, there are five multipaths corresponding to c. Leaving out R_c, the remaining four paths do not touch c. Hence, they are responsible for four dimensions inside the six-dimensional $T^1_X(0)$.

(6.4) Whenever $F < \Delta(Q)$ is a face, then there exist always degrees $R \in M = \pi^{-1}(\mathbb{Z} \cdot \theta^c)$ such that $R \leq 1$ on $\Delta(Q)$ and $R = 1$ exactly on F—just take R as the difference of $\underline{1}$ and an interior lattice point of the $\sigma^\vee$-face dual to F. In particular, as we have already seen in Theorem 29, every simple vertex b provides a contribution to T^1_X.

However, it might happen that simple vertices b do not provide multidegrees $R(b)$ of height 0. In the following *non-plane* flag quiver Q^3, every of the five inner vertices is simple in the sense of Definition 26. While the knots $c^1, \ldots, c^4$ provide multidegrees $R(c^i)$ of height 0, the knot b does not. The reason is that there is exactly one multipath leading from $\{c^3, c^4\}$ to $\{c^1, c^2\}$, but this multipath touches b.



Here is another example. The *plane* flag quiver Q^4 is built from Q^1 of Example 23 in (5.5) by adding one single rope. However, this procedure removes the b-contribution from $T^1_X(0)$, i.e. $T_0(b) = 0$.



In general, the absence of $T^1_X(0)$-pieces for a simple knot of Q means that the corresponding A_1-singularity is, even locally, not smoothable with deformations of degree 0. Hence, to obtain smoothability of $\mathbb{P}_{\nabla(Q)}$ in codimension three, a necessary condition is that $T_0(b) \neq 0$ for every simple knot b of Q. Using Theorem 31, this translates into the existence of detouring multipaths around every simple knot.

(6.5) We will show that the just mentioned necessary condition for smoothability in codimension three is sufficient, too. For simple knots b, the one-parameter deformations of $X_{\Delta(Q)}$ provided by the one-dimensional vector spaces $T^1_X(-R(b))$ are smoothings of the three-codimensional A_1-singularities along $Z(b)$. The latter are the orbits of the three-dimensional cones over the faces $F(Q\backslash b) \leq \Delta(Q)$. The question whether these one-parameter families fit together in a huge deformation doing all the smoothings simultaneously leads to the investigation of T^2_X.

Theorem 32. *Whenever $R \in M$ is a positive linear combination of degrees $R^i \in M$ carrying infinitesimal deformations, i.e. $T^1_X(-R^i) \neq 0$, then $T^2_X(-R) = 0$.*

Proof. Because of Corollary 14 and Theorem 1, we may assume that $R \leq 1$ on $\Delta(Q)$. To apply Theorem 2, we have first to check the three-dimensional faces of $\Delta(Q) \cap [R = 1] \subseteq \Delta(Q)$ for non-pyramids, i.e. to exclude octahedra and prisms corresponding to the triangular quivers depicted in (4.2). While the latter would provide a three-dimensional face with non-trivial D^1, which is excluded by Proposition 27, we need a finer argument for the octahedra:

The assumption of our theorem says that $R^i_\alpha = \langle a^\alpha, R^i\rangle \geq 1$ (in fact "$= 1$") holds exactly for the four arrows α containing the simple knot $b(R^i)$, cf. Corollary 28 or the proof of Theorem 31. In particular, since R is a positive linear combination of those R^i, the relation $\langle a^\alpha, R\rangle \geq 1$ is impossible, unless $t(\alpha)$ or $h(\alpha)$ is a simple vertex. On the other hand, if $R = 1$ was true on an octahedron $F(P)$, i.e. on the arrows of $\Gamma_Q(P) \cong \Gamma^{\mathrm{dbl}}(3)$, then two of the three vertices of $\Gamma_Q(P)$ would equal original vertices from $Q_0\backslash\{p^0, \ldots, p^l\}$. However, as for Q^2 in Example 23 of (6.1), the double arrow between these vertices implies that they cannot be simple—providing a contradiction.

Let us now check the remaining assumptions of Theorem 2. The two-dimensional, non-triangular faces of $\Delta(Q) \cap [R = 1]$ are squares provided by simple knots $b \in Q_0$; the four vertices of these squares correspond to the arrows containing b. In particular, the property of an arrow to contain exactly two knots translates into the property of an vertex a^α of $\Delta(Q)$ to belong to at most two of these squares. This means that we are done in case of $\dim(\Delta(Q) \cap [R = 1]) \geq 5$.

For the remaining case $\dim(\Delta(Q) \cap [R = 1]) = 4$, our argument requires a slight refinement. To obtain vanishing of D^2, we do not use Theorem 2 itself, but the stronger, original Theorem (4.7) of [2]: Since the quiver Q lacks oriented cycles, it provides a (non-linear) ordering of the set Q_0. Hence, whenever there is a connected set of squares in our face $[R = 1]$, then there is at least one vertex of one of these squares that contains only this single square. Now, beginning with this particular vertex, we may "clean" these squares in the sense of [2, (4.7)] successively. $\square$

Remark. It is not true in general that $T^2_X = 0$, cf. Example 5. However, the previous theorem says that at least the obstructions inside T^2_X are void.

Corollary 33. *Gorenstein singularities provided by flag quivers are smoothable in codimension three. Moreover, if every simple knot b can be by-passed with a multipath connecting its neighbors, then this can be done by a deformation of degree* 0.

Proof. With the notation of Corollary 28, we choose one element $R(b) \in M$ for each simple knot b. By the lack of obstructions, the corresponding one-parameter families fit into a common deformation over a smooth parameter space S.

Now, looking at the general points of the singular three-codimensional strata, S is obtained from their one-dimensional versal deformations via base change. In particular, for each of these strata, there is a hypersurface in S containing the parameters *not* smoothing this stratum. Hence, taking a curve inside S that avoids all these hypersurfaces outside $0 \in S$, yields the desired smoothing. $\square$

Example 34. The 5-dimensional projective varieties $\mathbb{P}_{\nabla(Q)}$ corresponding to the quivers Q^1 and Q^2 of (5.5) are smoothable in codimension three. On the other hand, for the quivers Q^3 and Q^4 of (6.4), we know this only for $X_{\Delta(Q)} = \mathrm{Cone}(\mathbb{P}_{\nabla(Q)})$ instead for $\mathbb{P}_{\nabla(Q)}$ itself.

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